Statistics and Probability Letters 178 (2021) 109181

Contents lists available at ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

A note on the stochastic domination condition and uniform integrability with applications to the strong law of large numbers

Andrew Rosalsky [a], Lê Văn Thành [b,*]

[a] Department of Statistics, University of Florida, Gainesville, FL 32611–8545, USA
[b] Department of Mathematics, Vinh University, 182 Le Duan, Vinh, Nghe An, Viet Nam

ARTICLE INFO

Article history:
Received 26 March 2021
Received in revised form 31 May 2021
Accepted 12 June 2021
Available online 23 June 2021

MSC:
28A25
60E15
60F15

Keywords:
Stochastic domination
Uniform integrability
De la Vallée Poussin criterion
Sequence of pairwise independent random variables
Strong law of large numbers

ABSTRACT

In this correspondence, we present new results concerning the concept of stochastic domination and apply them to obtain new results on uniform integrability and on the strong law of large numbers for sequences of pairwise independent random variables. Our result on the strong law of large numbers extends a result of Chen, Bai, and Sung (2014). The sharpness of the results is illustrated by three examples.

1. Introduction

Let I be a nonempty index set. In this note, we investigate the notion of a family of random variables $\{X_i, i \in I\}$ being *stochastically dominated* by a nonnegative random variable X. The usual definition of this notion is that

$$\sup_{i\in I} \mathbb{P}(|X_i| > x) \le \mathbb{P}(X > x), \text{ for all } x \in \mathbb{R}. \tag{1.1}$$

If the X_i, $i \in I$ are identically distributed, then (1.1) is of course satisfied with $X = |X_{i_0}|$ for any $i_0 \in I$. Many authors use an apparently weaker definition of $\{X_i, i \in I\}$ being stochastically dominated by a nonnegative random variable Y, namely that

$$\sup_{i\in I} \mathbb{P}(|X_i| > x) \le C_1\mathbb{P}(C_2Y > x), \text{ for all } x \in \mathbb{R} \tag{1.2}$$

for some constants $C_1, C_2 \in (0, \infty)$. It will be shown in Theorem 2.4, *inter alia*, that (1.1) and (1.2) are indeed equivalent.

* Corresponding author.
E-mail addresses: rosalsky@stat.ufl.edu (A. Rosalsky), levt@vinhuni.edu.vn (L.V. Thành).

https://doi.org/10.1016/j.spl.2021.109181

A sequence of random variables $\{X_n, n \geq 1\}$ is said to be *uniformly integrable* if

(i) $\sup_{n\geq 1} \mathbb{E}(|X_n|) < \infty$

and

(ii) for all $\varepsilon > 0$, there exists $\delta > 0$ such that for every event A with $\mathbb{P}(A) < \delta$,

$$\sup_{n\geq 1} \mathbb{E}(|X_n|\mathbf{1}(A)) < \varepsilon.$$

Here and thereafter, $\mathbf{1}(A)$ denotes the indicator function for set A. It is easy to see that (i) and (ii) are independent conditions in the sense that neither implies the other. Hu and Rosalsky (2011) showed that (ii) is indeed equivalent to the apparently stronger condition

(ii') for all $\varepsilon > 0$, there exists $\delta > 0$ such that for every sequence of events $\{A_n, n \geq 1\}$ with $\mathbb{P}(A_n) < \delta$, $n \geq 1$,

$$\sup_{n\geq 1} \mathbb{E}(|X_n|\mathbf{1}(A_n)) < \varepsilon.$$

The uniform integrability criterion (see, e.g., p. 94 in Chow and Teicher (1997)) asserts that a sequence of random variables $\{X_n, n \geq 1\}$ is uniformly integrable if and only if

$$\lim_{a\to\infty} \sup_{n\geq 1} \mathbb{E}(|X_n|\mathbf{1}(|X_n| > a)) = 0.$$

The following classical result of the renowned mathematician Charles de La Vallée Poussin (see, e.g., p. 19 in Meyer (1966)) provides another characterization of uniform integrability. We refer to it as the de La Vallée Poussin criterion for uniform integrability: A sequence of random variables $\{X_n, n \geq 1\}$ is uniformly integrable if and only if there exists a convex monotone function g defined on $[0, \infty)$ with $g(0) = 0$ such that

$$\lim_{x\to\infty} \frac{g(x)}{x} = \infty \quad \text{and} \quad \sup_{n\geq 1} \mathbb{E}(g(|X_n|)) < \infty.$$

The proof of the necessity half is far more difficult than that of the sufficiency half. On the other hand, the sufficiency half provides a very useful method for establishing uniform integrability of a sequence of random variables. For the sufficiency half, the condition that g is a convex monotone function defined on $[0, \infty)$ with $g(0) = 0$ is not needed; it can be weakened to the condition that g is a nonnegative measurable function defined on $[0, \infty)$.

Alternative proofs of the de La Vallée Poussin criterion for uniform integrability were provided by Chong (1979), Klenke (2014, p. 138), and Chandra (2015).

2. On stochastic domination and uniform integrability

For a family of random variables $\{X_i, i \in I\}$, the following theorem characterizes when the function

$$F(x) = 1 - \sup_{i\in I} \mathbb{P}(|X_i| > x), \ x \in \mathbb{R}$$

is the distribution function of a nonnegative random variable X such that $\{X_i, i \in I\}$ is stochastically dominated by X.

Theorem 2.1. *Let $\{X_i, i \in I\}$ be a family of random variables, and let*

$$F(x) = 1 - \sup_{i\in I} \mathbb{P}(|X_i| > x), \ x \in \mathbb{R}.$$

Then F is nondecreasing, right continuous and $\lim_{x\to-\infty} F(x) = 0$. Moreover, F is the distribution function of a nonnegative random variable X if and only if

$$\lim_{x\to\infty} F(x) = 1.$$

In such a case, $\{X_i, i \in I\}$ is stochastically dominated by X.

Proof. It is easy to see that $F(\cdot)$ is nondecreasing. Since $\mathbb{P}(|X_i| > x) = 1$ for all $i \in I$ and $x < 0$, we have $\lim_{x\to-\infty} F(x) = 0$. Let $G(x) = \sup_{i\in I} \mathbb{P}(|X_i| > x)$, $x \in \mathbb{R}$. To show that $F(\cdot)$ is right continuous, we will show that $G(\cdot)$ is right continuous; that is, we will show that

$$\lim_{x\to a^+} G(x) = G(a) \text{ for all } a \in \mathbb{R}.$$

Let $\varepsilon > 0$ and let $a \in \mathbb{R}$. Since $G(a) = \sup_{i\in I} \mathbb{P}(|X_i| > a)$, there exists $i_0 \in I$ such that

$$\mathbb{P}(|X_{i_0}| > a) > G(a) - \varepsilon/2.$$

Since the function

$$x \mapsto \mathbb{P}(|X_{i_0}| > x), \ x \in \mathbb{R}$$

is nonincreasing and right continuous, there exists $\delta > 0$ such that

$$-\varepsilon/2 < \mathbb{P}(|X_{i_0}| > x) - \mathbb{P}(|X_{i_0}| > a) \leq 0 \text{ for all } x \text{ such that } 0 \leq x - a < \delta.$$

Therefore, for x satisfying $0 \leq x - a < \delta$, we have

$$\begin{aligned} G(x) + \varepsilon &= \sup_{i \in I} \mathbb{P}(|X_i| > x) + \varepsilon \\ &\geq \mathbb{P}(|X_{i_0}| > x) + \varepsilon \\ &> \mathbb{P}(|X_{i_0}| > a) + \varepsilon/2 \\ &> G(a) \end{aligned}$$

and so $|G(x) - G(a)| < \varepsilon$. Thus $\lim_{x \to a^+} G(x) = G(a)$.

Since $F(\cdot)$ is nondecreasing, right continuous and $\lim_{x \to -\infty} F(x) = 0$, it is the distribution function of a random variable X if and only if $\lim_{x \to \infty} F(x) = 1$. From $\mathbb{P}(|X_i| > x) = 1$ for all $i \in I$ and $x < 0$, we have $F(x) = 0$ for all $x < 0$. Thus when $\lim_{x \to \infty} F(x) = 1$, $\mathbb{P}(X > x) = 1 - F(x) = 1$ for all $x < 0$ and so

$$\mathbb{P}(X \geq 0) = \mathbb{P}\left(\bigcap_{n=1}^{\infty} (X > -\frac{1}{n}) \right) = 1;$$

that is, $X \geq 0$ almost surely (a.s.). By the definition of F, it is clear that $\{X_i, i \in I\}$ is stochastically dominated by X. □

The following simple result is a direct consequence of Theorem 2.1.

Corollary 2.2. *Let $\{X_i, 1 \leq i \leq n\}$ be a finite family of random variables. Then $\{X_i, 1 \leq i \leq n\}$ is stochastically dominated by a nonnegative random variable X with distribution function*

$$F(x) = 1 - \sup_{1 \leq i \leq n} \mathbb{P}(|X_i| > x), \ x \in \mathbb{R}.$$

Before establishing the equivalence between the definitions of stochastic domination given in (1.1) and (1.2), we present the following simple lemma. This lemma is similar to Theorem 2.12.3 (i) of Gut (2013, p. 76).

Lemma 2.3. *Let $g : [0, \infty) \to [0, \infty)$ be a measurable function with $g(0) = 0$ which is bounded on $[0, A]$ and differentiable on $[A, \infty)$ for some $A \geq 0$. If ξ is a nonnegative random variable, then*

$$\mathbb{E}(g(\xi)) = \mathbb{E}(g(\xi)\mathbf{1}(\xi \leq A)) + g(A) + \int_A^{\infty} g'(x)\mathbb{P}(\xi > x)\mathrm{d}x. \tag{2.1}$$

Proof. Since $g(0) = 0$, we have

$$\begin{aligned} \mathbb{E}(g(\xi)) &= \mathbb{E}(g(\xi)\mathbf{1}(\xi \leq A)) + g(A) + \mathbb{E}\left(g(\xi)\mathbf{1}(\xi > A) - g(A)\right) \\ &= \mathbb{E}(g(\xi)\mathbf{1}(\xi \leq A)) + g(A) + \mathbb{E}\left(g(\xi\mathbf{1}(\xi > A)) - g(A)\right) \\ &= \mathbb{E}(g(\xi)\mathbf{1}(\xi \leq A)) + g(A) + \mathbb{E}\left(\int_A^{\xi\mathbf{1}(\xi > A)} g'(x)\mathrm{d}x\right) \\ &= \mathbb{E}(g(\xi)\mathbf{1}(\xi \leq A)) + g(A) + \mathbb{E}\left(\int_A^{\infty} g'(x)\mathbf{1}\left(\xi\mathbf{1}(\xi > A) > x\right)\mathrm{d}x\right) \\ &= \mathbb{E}(g(\xi)\mathbf{1}(\xi \leq A)) + g(A) + \int_A^{\infty} g'(x)\mathbb{P}(\xi\mathbf{1}(\xi > A) > x)\mathrm{d}x \text{ (by Fubini's theorem)} \\ &= \mathbb{E}(g(\xi)\mathbf{1}(\xi \leq A)) + g(A) + \int_A^{\infty} g'(x)\mathbb{P}(\xi > x)\mathrm{d}x \end{aligned}$$

proving (2.1). □

The next theorem establishes the equivalence between the definitions of stochastic domination given in (1.1) and (1.2).

Theorem 2.4. *Let $\{X_i, i \in I\}$ be a family of random variables. Then there exists a nonnegative random variable X satisfying (1.1) if and only if there exist a nonnegative random variable Y and constants $C_1, C_2 \in (0, \infty)$ satisfying (1.2). Moreover,*

(i) *if $g : [0, \infty) \to [0, \infty)$ is a measurable function with $g(0) = 0$ which is bounded on $[0, A]$ and differentiable on $[A, \infty)$ for some $A \geq 0$*

or

(ii) *if* $g : [0, \infty) \to [0, \infty)$ *is a continuous function which is eventually nondecreasing with* $\lim_{x\to\infty} g(x) = \infty$,

then the condition $\mathbb{E}(g(C_2Y)) < \infty$ *where* Y *is as in* (1.2) *implies that* $\mathbb{E}(g(X)) < \infty$ *where* X *is as in* (1.1).

Proof. The necessity half is immediate by taking $Y = X$ and $C_1 = C_2 = 1$.

Conversely, assume that there exist a nonnegative random variable Y and constants $C_1, C_2 \in (0, \infty)$ satisfying (1.2). Then

$$\lim_{x\to\infty} \sup_{i\in I} \mathbb{P}(|X_i| > x) \le C_1 \lim_{x\to\infty} \mathbb{P}(C_2Y > x) = 0$$

and so by Theorem 2.1, there exists a nonnegative random variable X with distribution function

$$F(x) = 1 - \sup_{i\in I} \mathbb{P}(|X_i| > x), \ \ x \in \mathbb{R}.$$

Thus

$$\sup_{i\in I} \mathbb{P}(|X_i| > x) = 1 - F(x) = \mathbb{P}(X > x), \ \ x \in \mathbb{R} \tag{2.2}$$

thereby verifying (1.1).

Next, we prove (i). By (1.2) and (2.2),

$$\mathbb{P}(X > x) = \sup_{i\in I} \mathbb{P}(|X_i| > x) \le C_1\mathbb{P}(C_2Y > x), \ \ x \in \mathbb{R}. \tag{2.3}$$

By Lemma 2.3 and (2.3), we have

$$\begin{aligned}
\mathbb{E}(g(X)) &= \mathbb{E}(g(X)\mathbf{1}(X \le A)) + g(A) + \int_A^\infty g'(x)\mathbb{P}(X > x)\mathrm{d}x \\
&\le \mathbb{E}(g(X)\mathbf{1}(X \le A)) + g(A) + C_1 \int_A^\infty g'(x)\mathbb{P}(C_2Y > x)\mathrm{d}x \\
&\le C + C_1\mathbb{E}(g(C_2Y)),
\end{aligned}$$

where C is a finite positive constant. Thus the condition $\mathbb{E}(g(C_2Y)) < \infty$ implies that $\mathbb{E}(g(X)) < \infty$ completing the proof of (i).

We now prove (ii). Let $A > 0$ be such that $g(A) > 0$ and g is nondecreasing on $[A, \infty)$. Let $B = \sup_{0\le x\le A} g(x)$. Then $B < \infty$ since g is continuous. Let

$$h(x) = \begin{cases} \dfrac{xg(A)}{A} & \text{if } \ 0 \le x < A \\ g(x) & \text{if } \ x \ge A \end{cases}$$

and

$$h^{-1}(t) = \inf\{x \ge 0 : h(x) \ge t\}, \ \ t \ge 0.$$

Note that for all $t \ge 0$ and $x \ge 0$,

$$t \le h(x) \ \text{ if and only if } \ h^{-1}(t) \le x.$$

It is easy to see that (2.3) ensures that

$$\mathbb{P}(X \ge x) \le C_1\mathbb{P}(C_2Y \ge x), \ \ x \in \mathbb{R}. \tag{2.4}$$

Now

$$\begin{aligned}
\mathbb{E}(g(X)) &= \int_0^\infty \mathbb{P}(g(X) \ge x)\mathrm{d}x \\
&= \int_0^\infty \mathbb{P}(g(X) \ge x, X < A)\mathrm{d}x + \int_0^\infty \mathbb{P}(g(X) \ge x, X \ge A)\mathrm{d}x \\
&= \int_0^B \mathbb{P}(g(X) \ge x, X < A)\mathrm{d}x + \int_0^\infty \mathbb{P}(h(X) \ge x, X \ge A)\mathrm{d}x \\
&\le B + \int_0^\infty \mathbb{P}(X \ge h^{-1}(x))\mathrm{d}x \\
&\le B + \int_0^\infty C_1\mathbb{P}(C_2Y \ge h^{-1}(x))\mathrm{d}x \text{ (by (2.4))} \\
&= B + \int_0^\infty C_1\mathbb{P}(h(C_2Y) \ge x)\mathrm{d}x \\
&= B + C_1\mathbb{E}(h(C_2Y)) \\
&= B + C_1\mathbb{E}(h(C_2Y)\mathbf{1}(C_2Y < A)) + C_1\mathbb{E}(h(C_2Y)\mathbf{1}(C_2Y \ge A)) \\
&\le B + C_1h(A) + C_1\mathbb{E}(g(C_2Y)).
\end{aligned}$$

Thus the condition $\mathbb{E}(g(C_2Y)) < \infty$ implies that $\mathbb{E}(g(X)) < \infty$ completing the proof of (ii). □

The next theorem generalizes Lemma 5.2.2 of Taylor (1978) and Lemma 3 of Wei and Taylor (1978) as well as the stronger form pointed out by Adler et al. (1992). In Theorem 2.5, it is shown that bounded moment type conditions on a family of random variables $\{X_i, i \in I\}$ can accomplish stochastic domination.

Throughout the rest of the paper, for $x \ge 0$, we let $\log(x)$ denote $\ln(\max\{\mathrm{e}, x\})$ where ln is the natural logarithm.

Theorem 2.5. *Let $\{X_i, i \in I\}$ be a family of random variables.*

(i) *Let $g : [0,\infty) \to [0,\infty)$ be a nondecreasing function with $\lim_{x\to\infty} g(x) = \infty$. If*

$$\sup_{i\in I} \mathbb{E}(g(|X_i|)) < \infty, \tag{2.5}$$

then there exists a nonnegative random variable X with distribution function $F(x) = 1 - \sup_{i\in I} \mathbb{P}(|X_i| > x),\ x \in \mathbb{R}$ such that $\{X_i, i \in I\}$ is stochastically dominated by X.

(ii) *If*

$$\sup_{i\in I} \mathbb{E}(|X_i|^p) < \infty \text{ for some } p > 0, \tag{2.6}$$

then there exists a nonnegative random variable X with distribution function $F(x) = 1 - \sup_{i\in I} \mathbb{P}(|X_i| > x),\ x \in \mathbb{R}$ such that $\{X_i, i \in I\}$ is stochastically dominated by X and

$$\mathbb{E}\left(X^p \log^{-1-\varepsilon}(X)\right) < \infty \text{ for all } \varepsilon > 0. \tag{2.7}$$

(iii) *If*

$$\sup_{i\in I} \mathbb{E}(|X_i|^p \log^{1+\varepsilon}(|X_i|)) < \infty \text{ for some } p > 0 \text{ and for some } \varepsilon > 0,$$

then there exists a nonnegative random variable X with distribution function $F(x) = 1 - \sup_{i\in I} \mathbb{P}(|X_i| > x),\ x \in \mathbb{R}$ such that $\{X_i, i \in I\}$ is stochastically dominated by X and

$$\mathbb{E}\left(X^p\right) < \infty.$$

Proof. (i) By the monotonicity of g and the Markov inequality, we have for all large x

$$\sup_{i\in I} \mathbb{P}(|X_i| > x) \le \frac{\sup_{i\in I} \mathbb{E}(g(|X_i|))}{g(x)}.$$

Thus by (2.5) and $\lim_{x\to\infty} g(x) = \infty$, we have

$$\lim_{x\to\infty} \sup_{i\in I} \mathbb{P}(|X_i| > x) \le \lim_{x\to\infty} \frac{\sup_{i\in I} \mathbb{E}(g(|X_i|))}{g(x)} = 0.$$

Then by applying Theorem 2.1, we get that $\{X_i, i \in I\}$ is stochastically dominated by a nonnegative random variable X with distribution function

$$F(x) = 1 - \sup_{i \in I} \mathbb{P}(|X_i| > x),\ x \in \mathbb{R}.$$

This completes the proof of (i).

(ii) It follows from (2.6) that (2.5) holds with $g(x) = x^p,\ x \geq 0$. Then by Part (i), $\{X_i, i \in I\}$ is stochastically dominated by a nonnegative random variable X with distribution function $F(x) = 1 - \sup_{i \in I} \mathbb{P}(|X_i| > x), x \in \mathbb{R}$. Let $\varepsilon > 0$ and let

$$h(x) = x^p \log^{-1-\varepsilon}(x),\ x \geq 0.$$

Then, we have

$$h'(x) = \frac{x^{p-1} \log^{\varepsilon}(x)\,(p\log(x) - (1+\varepsilon))}{\log^{2+2\varepsilon}(x)} \leq p x^{p-1} \log^{-1-\varepsilon}(x),\ x > \mathrm{e}. \tag{2.8}$$

By Lemma 2.3, (2.8), the Markov inequality, and (2.6), we have

$$\begin{aligned}
\mathbb{E}(h(X)) &= \mathbb{E}(h(X)\mathbf{1}(X \leq \mathrm{e})) + h(\mathrm{e}) + \int_{\mathrm{e}}^{\infty} h'(x)\mathbb{P}(X > x)\mathrm{d}x \\
&\leq 2\mathrm{e}^p + \int_{\mathrm{e}}^{\infty} p x^{p-1} \log^{-1-\varepsilon}(x)\mathbb{P}(X > x)\mathrm{d}x \\
&= 2\mathrm{e}^p + \int_{\mathrm{e}}^{\infty} p x^{p-1} \log^{-1-\varepsilon}(x) \sup_{i \in I} \mathbb{P}(|X_i| > x)\mathrm{d}x \\
&\leq 2\mathrm{e}^p + \int_{\mathrm{e}}^{\infty} p x^{-1} \log^{-1-\varepsilon}(x) \sup_{i \in I} \mathbb{E}(|X_i|^p)\mathrm{d}x \\
&= 2\mathrm{e}^p + p \sup_{i \in I} \mathbb{E}(|X_i|^p) \int_{\mathrm{e}}^{\infty} x^{-1} \log^{-1-\varepsilon}(x)\mathrm{d}x \\
&< \infty.
\end{aligned}$$

The proof of (ii) is completed.

(iii) The proof is similar to that of (ii) and is left to the reader. □

It easily follows from Exercise 6.2.8 in Chow and Teicher (1997, p. 183) that if a sequence of random variables $\{X_n, n \geq 1\}$ is stochastically dominated by a nonnegative random variable X with $\mathbb{E}(X^p) < \infty$ for some $p > 0$, then $\{|X_n|^p, n \geq 1\}$ is uniformly integrable. The next theorem is a partial converse of this result. We note that (2.9) is the necessary and sufficient condition for the Marcinkiewicz–Zygmund type weak law of large numbers when $0 < p \leq 1$ for sequences of independent and identically distributed random variables $\{X, X_n, n \geq 1\}$ (see, e.g., condition (4.2) of Theorem 6.4.2 in Gut (2013, p. 281)). When $1 \leq p < 2$, (2.9) is the sufficient condition for a Marcinkiewicz–Zygmund type weak law of large numbers for double arrays of independent random variables which are stochastically dominated by a nonnegative random variable X (see Theorem 3.2 in Rosalsky and Thanh (2009)).

Theorem 2.6. *Let $p > 0$ and let $\{X_n, n \geq 1\}$ be a sequence of random variables. If $\{|X_n|^p, n \geq 1\}$ is uniformly integrable, then there exists a nonnegative random variable X with distribution function $F(x) = 1 - \sup_{n \geq 1} \mathbb{P}(|X_n| > x), x \in \mathbb{R}$ such that $\{X_n, n \geq 1\}$ is stochastically dominated by X,*

$$\lim_{n \to \infty} n\mathbb{P}\left(X > n^{1/p}\right) = 0, \tag{2.9}$$

and

$$\mathbb{E}\left(X^p \log^{-1-\varepsilon}(X)\right) < \infty \text{ for all } \varepsilon > 0. \tag{2.10}$$

Proof. By the de La Vallée Poussin criterion for uniform integrability, there exists a nondecreasing function g defined on $[0, \infty)$ with $g(0) = 0$ such that

$$\lim_{x \to \infty} \frac{g(x)}{x} = \infty, \tag{2.11}$$

and

$$\sup_{i \geq 1} \mathbb{E}(g(|X_i|^p)) < \infty. \tag{2.12}$$

Now by Theorem 2.5 (i), (2.12) implies that $\{X_n, n \geq 1\}$ is stochastically dominated by a nonnegative random variable X with distribution function

$$F(x) = 1 - \sup_{i \geq 1} \mathbb{P}(|X_i| > x),\ x \in \mathbb{R}.$$

We thus have by the Markov inequality that

$$\begin{aligned}\lim_{n\to\infty} n\mathbb{P}\left(X > n^{1/p}\right) &= \lim_{n\to\infty} n\sup_{i\ge 1}\mathbb{P}(|X_i|^p > n)\\ &\le \lim_{n\to\infty} n\sup_{i\ge 1}\mathbb{P}(g(|X_i|^p) \ge g(n))\\ &\le \lim_{n\to\infty} n\sup_{i\ge 1}\frac{\mathbb{E}(g(|X_i|^p))}{g(n)}\\ &= \sup_{i\ge 1}\mathbb{E}(g(|X_i|^p))\lim_{n\to\infty}\frac{n}{g(n)} = 0\end{aligned}$$

by (2.11) and (2.12) thereby proving (2.9).

Finally, the uniform integrability hypothesis ensures that (2.6) holds and so (2.10) follows from Theorem 2.5 (ii). □

3. An application of Theorem 2.5 to the strong law of large numbers for pairwise independent random variables

In this section, we present two strong laws of large numbers (SLLNs). They are a consequence of Theorem 2.5 and the following proposition which is proved along the lines of Theorem 1 of Thành (2020).

Proposition 3.1. *Let $1 \le p < 2$ and let $\{X_n, n \ge 1\}$ be a sequence of pairwise independent random variables which is stochastically dominated by a nonnegative random variable X satisfying*

$$\mathbb{E}\left(X^p \log^\gamma(X)\right) < \infty \text{ for some } \gamma \in \mathbb{R}.$$

If $p = 1$, assume that $\gamma \ge 0$. Then for all $\alpha \ge 1/p$, we have

$$\sum_{n=1}^\infty n^{\alpha p-2}\mathbb{P}\left(\max_{1\le k\le n}\left|\sum_{i=1}^k (X_i - \mathbb{E}(X_i))\right| > \varepsilon n^\alpha \log^{-\gamma}(n)\right) < \infty \text{ for all } \varepsilon > 0$$

and, a fortiori, the SLLN

$$\lim_{n\to\infty}\frac{\sum_{i=1}^n (X_i - \mathbb{E}(X_i))}{n^{1/p}\log^{-\gamma}(n)} = 0 \text{ a.s.}$$

is obtained.

In Theorem 3.6 of Chen et al. (2014), it is proved that if $\{X_n, n \ge 1\}$ is a sequence of pairwise independent and identically distributed random variables with $\mathbb{E}(X_1) = 0$ and $\mathbb{E}(|X_1|^p \log^\gamma(|X_1|)) < \infty$ for some $1 < p < 2$ and for some $p < \gamma < 2$, then

$$\sum_{n=1}^\infty n^{-1}\mathbb{P}\left(\max_{1\le k\le n}\left|\sum_{i=1}^k X_i\right| > \varepsilon n^{1/p}\right) < \infty \text{ for all } \varepsilon > 0.$$

Therefore, both Proposition 3.1 and the following theorem are stronger results than Theorem 3.6 of Chen et al. (2014).

Theorem 3.2. *Let $1 \le p < 2$, and let $\{X_n, n \ge 1\}$ be a sequence of pairwise independent random variables satisfying*

$$\sup_{n\ge 1}\mathbb{E}\left(|X_n|^p \log^\gamma(|X_n|)\right) < \infty \quad \text{for some } \gamma > 1. \tag{3.1}$$

Then for all $\alpha \ge 1/p$, we have

$$\sum_{n=1}^\infty n^{\alpha p-2}\mathbb{P}\left(\max_{1\le k\le n}\left|\sum_{i=1}^k (X_i - \mathbb{E}(X_i))\right| > \varepsilon n^\alpha\right) < \infty \text{ for all } \varepsilon > 0$$

and, a fortiori, the SLLN

$$\lim_{n\to\infty}\frac{\sum_{i=1}^n (X_i - \mathbb{E}(X_i))}{n^{1/p}} = 0 \text{ a.s.} \tag{3.2}$$

is obtained.

Proof. By Part (iii) of Theorem 2.5 and (3.1), there exists a nonnegative random variable X with $\mathbb{E}(X^p) < \infty$ such that $\{X_n, n \ge 1\}$ is stochastically dominated by X. Theorem 3.2 then follows from Proposition 3.1. □

Similarly, by Part (ii) of Theorem 2.5 and Proposition 3.1, we have the following SLLN.

Theorem 3.3. *Let $1 < p < 2$, and let $\{X_n, n \geq 1\}$ be a sequence of pairwise independent random variables satisfying*

$$\sup_{n\geq 1} \mathbb{E}\left(|X_n|^p\right) < \infty.$$

Then for all $\alpha \geq 1/p$ and for all $\gamma > 1$, we have

$$\sum_{n=1}^{\infty} n^{\alpha p-2}\mathbb{P}\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k}(X_i - \mathbb{E}(X_i))\right| > \varepsilon n^{\alpha}\log^{\gamma}(n)\right) < \infty \text{ for all } \varepsilon > 0$$

and, a fortiori, the SLLN

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n}(X_i - \mathbb{E}(X_i))}{n^{1/p}\log^{\gamma}(n)} = 0 \text{ a.s.}$$

is obtained.

4. Three interesting examples

In this section, we will present three interesting examples to illustrate the sharpness of the results. The first example shows that, in Part (ii) of Theorem 2.5, (2.7) cannot be strengthened to

$$\mathbb{E}(X^p \log^{-1}(X)) < \infty. \tag{4.1}$$

It also shows that the assumption $\{|X_n|^p, n \geq 1\}$ being uniformly integrable in Theorem 2.6 cannot be weakened to $\sup_{n\geq 1}\mathbb{E}(|X_n|^p) < \infty$.

Example 4.1. Let $p > 0$ and $\{X_n, n \geq 1\}$ be a sequence of random variables such that

$$\mathbb{P}(X_n = 0) = 1 - \frac{1}{n},\ \mathbb{P}(X_n = n^{1/p}) = \frac{1}{n},\ n \geq 1.$$

Then $\sup_{n\geq 1}\mathbb{E}(|X_n|^p) = 1$ and so (2.6) holds. By Part (ii) of Theorem 2.5, $\{X_n, n \geq 1\}$ is stochastically dominated by a nonnegative random variable X with distribution function $F(x) = 1 - \sup_{n\geq 1}\mathbb{P}(X_n > x), x \in \mathbb{R}$, and (2.7) holds.

We will now show that (4.1) fails. For $x \in \mathbb{R}$, let $\lceil x\rceil$ be the smallest integer that is greater than x. Then

$$\mathbb{P}(X > x) = \begin{cases} 1 & \text{if } x < 1,\\ \dfrac{1}{\lceil x^p\rceil} & \text{if } x \geq 1.\end{cases} \tag{4.2}$$

Set $g(x) = x^p\log^{-1}(x),\ x \geq 0$. Then

$$g'(x) = px^{p-1}\log^{-1}(x) - x^{p-1}\log^{-2}(x) = x^{p-1}\log^{-1}(x)(p - \log^{-1}(x)) > px^{p-1}\log^{-1}(x)/2$$

for all large x. By Lemma 2.3 and (4.2), for all A large enough, we have

$$\begin{aligned}\mathbb{E}(X^p\log^{-1}(X)) &\geq \frac{p}{2}\int_A^{\infty} x^{p-1}\log^{-1}(x)\mathbb{P}(X > x)\mathrm{d}x\\ &= \frac{p}{2}\int_A^{\infty}\frac{x^{p-1}\log^{-1}(x)}{\lceil x^p\rceil}\mathrm{d}x\\ &= \infty\end{aligned}$$

and so (4.1) fails. Now, for all $a > 0$,

$$\sup_{n\geq 1}\mathbb{E}(|X_n|^p\mathbf{1}(|X_n| > a)) = 1.$$

Therefore, $\lim_{a\to\infty}\sup_{n\geq 1}\mathbb{E}(|X_n|^p\mathbf{1}(|X_n > a)) = 1$ and so $\{|X_n|^p, n \geq 1\}$ is not uniformly integrable. We also have from (4.2) that

$$\lim_{n\to\infty} n\mathbb{P}(X > n^{1/p}) = \lim_{n\to\infty}\frac{n}{n+1} = 1,$$

and thus (2.9) fails.

The second example shows that in Theorem 2.6, (2.10) cannot be strengthened to (4.1).

Example 4.2. Let $\{X_n, n \geq 1\}$ be a sequence of random variables such that

$$\mathbb{P}(X_n = 0) = 1 - \frac{1}{n},\ \mathbb{P}\left(X_n = n\left(\log\left(\log(n)\right)\right)^{-1}\right) = \frac{1}{n},\ n \geq 1.$$

Then $\sup_{n\geq 1}\mathbb{E}(|X_n|\log(\log(|X_n|))) < \infty$ and so $\{X_n, n \geq 1\}$ is uniformly integrable by the de La Vallée Poussin criterion. The hypotheses of Theorem 2.6 are satisfied with $p = 1$. Then by Theorem 2.6, $\{X_n, n \geq 1\}$ is stochastically dominated by a nonnegative random variable X with distribution function $F(x) = 1 - \sup_{n\geq 1}\mathbb{P}(X_n > x)$, $x \in \mathbb{R}$. Then

$$\mathbb{P}(X > x) = \begin{cases} 1 & \text{if } x < 1, \\ \dfrac{1}{\alpha_x} & \text{if } x \geq 1, \end{cases}$$

where for $x \geq 1$, α_x is the smallest integer n such that $n\,(\log(\log(n)))^{-1} > x$. Then for $x \geq 1$,

$$\alpha_x\,(\log(\log(\alpha_x)))^{-1} > x \geq (\alpha_x - 1)\,(\log(\log(\alpha_x - 1)))^{-1} \tag{4.3}$$

and it follows from (4.3) that

$$\alpha_x \sim x\log(\log(x)) \text{ as } x \to \infty.$$

Proceeding as in Example 4.1, for all A large enough, we have

$$\begin{aligned}\mathbb{E}(X\log^{-1}(X)) &\geq \frac{1}{2}\int_A^\infty \log^{-1}(x)\mathbb{P}(X > x)\mathrm{d}x \\ &= \frac{1}{2}\int_A^\infty \frac{\log^{-1}(x)}{\alpha_x}\mathrm{d}x \\ &\geq \frac{1}{4}\int_A^\infty \frac{1}{x(\log(x))\log(\log(x))}\mathrm{d}x \\ &= \infty\end{aligned}$$

and so (4.1) fails.

The third example illustrates the sharpness of Theorem 3.2. If $1 \leq p < 2$ and $\{X_n, n \geq 1\}$ is a sequence of pairwise independent and identically distributed integrable random variables, then the necessary and sufficient condition for the SLLN (3.2) is $\mathbb{E}(|X_1|^p) < \infty$ (see Corollary 2 in Thành (2020)). However, the following example shows that in Theorem 3.2, the SLLN (3.2) may fail if (3.1) is weakened to

$$\sup_{n\geq 1}\mathbb{E}\left(|X_n|^p\log(|X_n|)\right) < \infty. \tag{4.4}$$

Example 4.3. Let $1 \leq p < 2$ and $\{X_n, n \geq 1\}$ be a sequence of independent random variables such that

$$\mathbb{P}(X_n = 0) = 1 - \frac{1}{(n+1)\log(n+1)},\ \mathbb{P}\left(X_n = \pm(n+1)^{1/p}\right) = \frac{1}{2(n+1)\log(n+1)},\ n \geq 1.$$

Then $\sup_{n\geq 1}\mathbb{E}(|X_n|^p\log^\gamma(|X_n|)) = \infty$ for all $\gamma > 1$ but $\sup_{n\geq 1}\mathbb{E}(|X_n|^p\log(|X_n|)) = 1/p$ and so (3.1) fails but (4.4) holds.

Now if the SLLN (3.2) holds, then

$$\lim_{n\to\infty}\frac{X_n}{n^{1/p}} = 0 \text{ a.s.} \tag{4.5}$$

Since the sequence $\{X_n, n \geq 1\}$ is comprised of independent random variables, the Borel–Cantelli lemma and (4.5) ensure that

$$\sum_{n=1}^\infty \mathbb{P}(|X_n| > n^{1/p}) < \infty. \tag{4.6}$$

However, we have

$$\sum_{n=1}^\infty \mathbb{P}(|X_n| > n^{1/p}) = \sum_{n=1}^\infty \frac{1}{(n+1)\log(n+1)} = \infty$$

contradicting (4.6). Therefore, the SLLN (3.2) must fail.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgements

The authors are grateful to the Referees for carefully reading the manuscript and for offering useful comments and suggestions which enabled them to improve the paper.

Funding

This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

References

Adler, A., Rosalsky, A., Taylor, R.L., 1992. Some strong laws of large numbers for sums of random elements. Bull. Inst. Math. Acad. Sin. 20 (4), 335–357.
Chandra, T.K., 2015. De La Vallée Poussin's theorem, uniform integrability, tightness and moments. Statist. Probab. Lett. 107, 136–141.
Chen, P., Bai, P., Sung, S.H., 2014. The von Bahr–Esseen moment inequality for pairwise independent random variables and applications. J. Math. Anal. Appl. 419 (2), 1290–1302.
Chong, K.-M., 1979. On a theorem concerning uniform integrability. Publ. Inst. Math. (Beograd) (N.S.) 25 (39), 8–10.
Chow, Y.S., Teicher, H., 1997. Probability Theory: Independence, Interchangeability, Martingales, third ed. Springer–Verlag, New York.
Gut, A., 2013. Probability: A Graduate Course, second ed. Springer, New York.
Hu, T.-C., Rosalsky, A., 2011. A note on the de La Vallée Poussin criterion for uniform integrability. Statist. Probab. Lett. 81 (1), 169–174.
Klenke, A., 2014. Probability Theory: A Comprehensive Course, second ed. Springer–Verlag, London.
Meyer, P.A., 1966. Probability and Potentials. Blaisdell Publishing Company, Ginn and Company, Waltham, Massachusetts, Toronto, Ont.–London.
Rosalsky, A., Thanh, L.V., 2009. Weak laws of large numbers for double sums of independent random elements in Rademacher type p and stable type p Banach spaces. Nonlinear Anal. TMA 71 (12), e1065–e1074.
Taylor, R.L., 1978. Stochastic Convergence of Weighted Sums of Random Elements in Linear Spaces. Lecture Notes in Mathematics, vol. 672, Springer, Berlin.
Thành, L.V., 2020. On the Baum–Katz theorem for sequences of pairwise independent random variables with regularly varying normalizing constants. C. R. Math. Acad. Sci. Paris 358 (11–12), 1231–1238.
Wei, D., Taylor, R.L., 1978. Convergence of weighted sums of tight random elements. J. Multivariate Anal. 8 (2), 282–294.